A rational function approximation of the singular eigenfunction of the monoenergetic neutron transport equation

1984 J. Phys. A: Math. Gen. 17 2743

(http://iopscience.iop.org/0305-4470/17/14/018)

J. Phys. A: Math. Gen. 17 (1984) 2743–2758. Printed in Great Britain

A rational function approximation of the singular eigenfunction of the monoenergetic neutron transport equation

A Sengupta

Nuclear Engineering and Technology Programme, Indian Institute of Technology Kanpur, India.

Received 2 February 1984

Abstract. In this paper we demonstrate how a proper rational fraction approximation to the singular eigenfunction of the neutron transport theory can be constructed based on the properties of generalised functions and singular integral equations. The parameters of the approximant are determined by a proper use of the orthogonality integrals satisfied by the Case eigenfunctions. This ensures the convergence of the approximant to its exact singular distributional form. Use of Lebesgue integrable spaces made in the analysis leads to a new possibility of approximating functions in L_p, $1<p<\infty$, and also of finding approximate solutions of singular integral equations.

1. Introduction

Orthogonal polynomials, which are the solutions of certain second-order linear differential equations with variable coefficients, have played a significant role in the approximation of functions. This is because these polynomials are the best approximations to zero (that is they are the polynomials of minimum norm) in their respective function spaces. Thus in L_2 and L_∞ the Legendre and Chebyshev polynomials are, respectively, the polynomials of best approximation in $(-1, 1)$, which means that an arbitrary function in these spaces can be best approximated in a compact subspace by some linear combination of the respective polynomials.

In contrast to the second-order linear differential equations and the classical orthogonal polynomials they generate, the integrodifferential equation of monoenergetic neutron transport theory gives rise to a maximal (complete) set of non-polynomial eigenfunctions such that any arbitrary Holder continuous function in the interval $(-1, 1)$, or the half interval $(0, 1)$, properly bounded at the end points, can be represented as a linear combination of them (Case and Zweifel 1967, Roos 1969). These eigenfunctions are those of the operator (Larsen and Habetler 1973)

$$\mathbb{T}\cdot=\mu\cdot+\frac{c}{2(1-c)}\int_{-1}^{1}\mathrm{d}\mu\,\mu\cdot \tag{1}$$

where c is an arbitrary constant that we take less than unity. This operator is the inverse of the operator arising in the solution of the monoenergetic neutron transport equation i.e., of (Larsen and Habetler 1973)

$$\mathbb{T}_0\cdot=\frac{1}{\mu}\cdot-\frac{c}{2\mu}\int_{-1}^{1}\mathrm{d}\mu\cdot.$$

0305-4470/84/142743 + 16\$02.25

The operators $\mathbb{T}$ and $\mathbb{T}_0$ satisfy the eigenvalue equations

$$\mathbb{T}_0\phi(\nu,\mu)=(1/\nu)\phi(\nu,\mu)$$

$$\mathbb{T}\phi(\nu,\mu)=\nu\phi(\nu,\mu)$$

where

$$\phi(\nu,\mu)=\frac{c\nu}{2}\frac{1}{\nu-\mu}, \qquad \nu\notin(-1,1) \tag{2a}$$

$$\phi(\nu,\mu)=\frac{c\nu}{2}P\frac{1}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu), \qquad \nu\in(-1,1) \tag{2b}$$

$$\int_{-1}^{1}\phi(\nu,\mu)\,\mathrm{d}\mu=1 \tag{2c}$$

so that

$$\lambda(\nu)=1-\frac{c\nu}{2}\ln\frac{1+\nu}{1-\nu} \tag{2d}$$

Note that equation (2b) is the general distributional solution of the equation

$$(\nu-\mu)\phi(\nu,\mu)=\tfrac{1}{2}c\nu$$

with $\lambda(\nu)$ an arbitrary function, and one of our main motivations in this paper is to obtain a rational function approximation of it.

It is a simple matter to obtain a spectral resolution of $\mathbb{T}$. The point spectrum of $\mathbb{T}$, $P\sigma(\mathbb{T})$, consists of those values of ν for which $(\mathbb{T}-\nu\mathbb{I})$ is not 1:1. Then it is not difficult to see that ν must satisfy the equation

$$\frac{c\nu}{2}\ln\frac{\nu+1}{\nu-1}=1, \qquad \nu\notin(-1,1)$$

having two real roots $\pm\nu_0$. Thus $P\sigma(\mathbb{T})=\pm\nu_0$. Also, since $\mathbb{T}$ is the sum of the self adjoint multiplication operator μ and the μ-compact integral operator $c/2(1-c)\int_{-1}^{1}\mathrm{d}\mu\mu\cdot$, the continuous spectrum of $\mathbb{T}$, $C\sigma(\mathbb{T})$, is that of μ, i.e., $C\sigma(\mathbb{T})=(-1,1)$. Therefore $\mathbb{T}$ has the spectral resolution

$$\mathbb{T}=\nu_0\mathbb{P}(\nu_0)-\nu_0\mathbb{P}(-\nu_0)+\int_{-1}^{1}\mathrm{d}\nu\,\nu\mathbb{P}(\nu)$$

where the ranges of the projection operators $\mathbb{P}$ are

$$\mathscr{R}(\mathbb{P}(\pm\nu_0))=\mathrm{Sp}\{\phi(\pm\nu_0),\ \nu_0\notin(-1,1)\}$$

$$\mathscr{R}(\mathbb{P}(\nu))=\mathrm{Sp}\{\phi(\nu),\ \nu\in(-1,1)\}.$$

These projection operators are orthogonal i.e., $\mathscr{R}(\mathbb{P}(\nu))\perp\mathscr{R}(\mathbb{P}(\nu'))$ $\nu\neq\nu'$. Hence $\mathscr{D}(\mathbb{T})$, the domain of $\mathbb{T}$, can be decomposed as

$$\mathscr{D}(\mathbb{T})=\mathscr{R}(\mathbb{P}(\nu_0))\oplus\mathscr{R}(\mathbb{P}(-\nu_0))\oplus\mathscr{R}(\mathbb{P}(\nu))$$

which implies that if $f(\mu)\in\mathscr{D}(\mathbb{T})$, then

$$f(\mu)=a_{0+}\phi(\nu_0,\mu)+a_{0-}\phi(-\nu_0,\mu)+\int_{-1}^{1}A(\nu)\phi(\nu,\mu)\,\mathrm{d}\nu. \tag{3}$$

In the classical formulation of equation (3) (Case and Zweifel 1967), $\mathscr{D}(\mathbb{T})$ is the space of Holder continuous functions on $(-1, 1)$ or $(0, 1)$, with proper boundedness properties at the end points, that is the space of Holder continuous functions in the extended sense. This restriction on $\mathscr{D}(\mathbb{T})$ is imposed by the theory of Hilbert boundary value problems and the theory of singular integral equations (Roos 1969). However the same results hold if $f(\mu) \in L_p$, $1 < p < \infty$, as shown in § 2 below. In fact, a major objective of § 2 is to demonstrate this possibility.

The discussion above indicates that it is possible to use the eigenfunctions $\{\phi(\pm\nu_0, \mu), \phi(\nu, \mu), -1 < \nu < 1\}$ as a complete set in representing an arbitrary function in L_p, $1 < p < \infty$. The advantage of doing this is that unlike in an infinite series that the use of classical orthogonal polynomials entails, equation (3) does not involve any convergence arguments and the expression on the right is a pointwise representation of the function on the left. The disadvantage of equation (3) is that it is a singular integral equation for the expansion coefficient $A(\nu)$. This equation can be solved by converting it to a Hilbert boundary value problem (Roos 1969) or by numerical means (Dow and Elliot 1979, Elliot 1982). The $A(\nu)$ will, however, involve the normalisation factor,

$$N(\nu) = \nu / g(c, \nu), \qquad g^{-1}(c, \nu) = \lambda^2(\nu) + \tfrac{1}{4}\pi^2 c^2 \nu^2$$

because of the orthogonality of $\phi(\nu, \mu)$ (Case and Zweifel 1967), and further evaluation of the integral $\int_{-1}^{1} A(\nu)\phi(\nu, \mu)\mathrm{d}\nu$, to reconstruct the function from its expansion coefficients becomes difficult. The reconstruction may be achieved by, for example, expanding $A(\nu)$ in a power series in ν. In its use in neutron transport theory, the integrand $A(\nu)\phi(\nu, \mu)$ is to be further multiplied by $\exp(-x/\nu)$ before this integration is carried out, thus compounding the difficulty. This constitutes a practical drawback in the use of the Case eigenfunctions as a (topological) basis for functions in L_p and suggests the need for an approximation procedure that replaces the integral in equation (3) by a sum i.e., a procedure that discretises $\nu \in (-1, 1)$. The motivation of this crucial step is contained in the argument that any finite interval of the real line is separable in the usual metric because the rationals form a denumerable dense subset of $\mathbb{R}$.

In this paper we demonstrate that it is possible, using the notion of equivalent Cauchy sequences in the theory of distributions, to find a proper rational function approximation of the singular eigenfunction of monoenergetic neutron transport theory i.e., of equation (2*b*), leading to a discretisation of $\nu \in (-1, 1)$. Section 2 develops the necessary mathematical framework, which is then applied to the problem of finding the rational function approximation in § 3. Our arguments in this section are also based on the ideas introduced by Sengupta (1982). In order to gain a proper perspective of the method, we discuss in § 4 three different ways by which the unknown coefficients of a rational function approximant may be obtained and illustrate their use with the simple regular function $\exp(x)$. The example shows that the method adapted by us has advantages over the other two, and provides a necessary support for the method.

2. Mathematical formulation

We present in this section, various results in the theory of the Hilbert boundary value problem and singular integral equations that are necessary to further development. Our basic aim is to examine this theory from the view point of distributions, or generalised functions, and to demonstrate how Cauchy sequences in a suitable space of (test) functions can be used to formulate our approximation method. The main results of this section are: (a) Theorems 2 and 4 as applied to a finite interval. These theorems allow us to give a precise meaning to the convergence, as also the motivation, of a rational function approximation of the singular eigenfunction, equation (2*b*), constructed in § 3 and (b) to show that it is possible to consider the domain of $\mathbb{T}$ to be functions in L_p, and thereby to conclude that equation (3) holds for L_p, $1<p<\infty$. We do not give any proof of the stated results and provide only the essential discussions. The details may be found in standard references, for example in the books by Sadosky (1979), Korevaar (1968) and Titchmarsh (1937).

The Hilbert boundary value problem for a finite open contour is the following. Find a sectionally analytic function $\Phi(z)$, $z=x+\mathrm{i}\varepsilon$, which has as its boundary a open contour Γ; specifically we consider $\Gamma=(a, b)$ on the real line. $\Phi(z)$ is to be suitably bounded as z tends to the end points a and b and to vanish as $|z|\to\infty$. The boundary conditions to be satisfied by $\Phi(z)$ on Γ is

$$\Phi^+(x)=A(x)\Phi^-(x)+B(x), \qquad x\in(a, b) \tag{4}$$

where $A(x)$ and $B(x)$ are arbitrary Helder continuous functions on (a, b) and $A(x)$ is, in addition, non-vanishing on Γ. A related equivalent problem is that of a singular integral equation (SIE) with Cauchy kernel. Such an equation can be obtained from equation (4), or equivalently equation (4) can be transformed into a SIE. Thus, consider a function $\phi(x)$, Holder continuous in the extended sense on (a, b), for the sectionally analytic function

$$\Phi(z)=\frac{1}{2\pi\mathrm{i}}\int_a^b \frac{\phi(x')}{x'-z}\,\mathrm{d}x', \qquad z=x+\mathrm{i}\varepsilon \tag{5}$$

with boundary values on (a, b) expressed as

$$\Phi^+(x)=\lim_{\varepsilon\to 0}\Phi(x+\mathrm{i}\varepsilon); \qquad \Phi^-(x)=\lim_{\varepsilon\to 0}\Phi(x-\mathrm{i}\varepsilon),$$

where we always write $\varepsilon\to 0$ to mean $\varepsilon\to 0+$. Now using the Plemelj formulae for the boundary values,

$$\Phi^+(x)=\tfrac{1}{2}\phi(x)+\frac{1}{2\pi\mathrm{i}}⨍_a^b \frac{\phi(x')}{x'-x}\,\mathrm{d}x' \tag{6a}$$

$$\Phi^-(x)=-\tfrac{1}{2}\phi(x)+\frac{1}{2\pi\mathrm{i}}⨍_a^b \frac{\phi(x')}{x'-x}\,\mathrm{d}x' \tag{6b}$$

or, equivalently the equations,

$$\Phi^+(x)-\Phi^-(x)=\phi(x)$$

$$\Phi^+(x)+\Phi^-(x)=\frac{1}{\pi\mathrm{i}}⨍_a^b \frac{\phi(x')}{x'-x}\,\mathrm{d}x'$$

we get from equation (4) the SIE for $\phi(x)$

$$\alpha(x)\phi(x)+\beta(x)\frac{1}{2\pi\mathrm{i}}⨍_a^b \frac{\phi(x')}{x'-x}\,\mathrm{d}x'=B(x). \tag{7}$$

Here $⨍$ denotes a principal value integral and

$$\alpha(x)=\tfrac{1}{2}(1+A(x)), \qquad \beta(x)=1-A(x)$$

are Holder continuous functions. The relevance of SIE in monoenergetic neutron transport theory arises from the fact that equation (3) leads to such an equation for the expansion coefficient $A(\nu)$.

Now consider $\Gamma = \mathbb{R}$, the real line. Then in the integrals above, the limits can be replaced by $-\infty$ and ∞, and the Hilbert problem is simplified as it is no longer necessary to specify the end conditions on Γ separately. Hence if in equation (5) we decompose $-\mathrm{i}/(x'-z)$ into its real and imaginary parts to get

$$\frac{1}{x'-z}=\frac{x'-x}{(x'-x)^2+\varepsilon^2}+\mathrm{i}\frac{\varepsilon}{(x'-x)^2+\varepsilon^2}, \tag{8}$$

we have

$$\begin{aligned}\Phi(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon}{(x-x')^2+\varepsilon^2}\phi(x')\,\mathrm{d}x'+\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{x-x'}{(x-x')^2+\varepsilon^2}\phi(x')\,\mathrm{d}x\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\delta_\varepsilon(x-x')\phi(x')\,\mathrm{d}x'+\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}P_{0\varepsilon}(x-x')\phi(x')\,\mathrm{d}x'\\ &\equiv \tfrac{1}{2}U(x,\varepsilon)+\tfrac{1}{2}\mathrm{i}V(x,\varepsilon)\end{aligned}$$

where, by definition,

$$\delta_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2},\qquad P_{0\varepsilon}(x)=\frac{1}{\pi}\frac{x}{x^2+\varepsilon^2},$$

and

$$U(x,\varepsilon)=\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{\phi(x')}{(x'-x)^2+\varepsilon^2}\,\mathrm{d}x'$$

$$V(x,\varepsilon)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x'-x}{(x'-x)^2+\varepsilon^2}\phi(x')\,\mathrm{d}x',$$

the real and imaginary parts of the Cauchy integral representation of $\Phi(z)$ with density function $\phi(x)$, are known as the Poisson and conjugate Poisson integrals of ϕ generated respectively by the kernels δ_ε and $P_{0\varepsilon}$ of the operator $\mathbb{K}:\phi\to\mathbb{K}\phi=k*\phi$ where

$$k*\phi=\int_{-\infty}^{\infty}k(x-x')\phi(x')\,\mathrm{d}x'. \tag{9}$$

We now introduce (Korevaar 1968) those concepts from the theory of generalised functions necessary in our further development.

Definition 1. The space of test functions $\mathscr{D}$, consists of all real valued functions φ that are infinitely smooth and zero outside some fixed finite interval J with convergence in $\mathscr{D}$ defined in the following way. $\varphi_j\to\varphi\in\mathscr{D}$ if (a) the supporting intervals of φ_j and φ belong to a fixed sub-interval $I\subset J$, and (b) $\varphi_j^{(m)}\to\varphi^{(m)}$ uniformly on I for all m and j. Therefore $\varphi\in\mathscr{D}(I)\Rightarrow\varphi\in C_0^\infty(I)$.

The common example of a test function normalised to unity is

$$\varphi_\alpha(x)=\begin{cases}c_\alpha\exp\left(-\dfrac{\alpha^2}{\alpha^2-x^2}\right), & |x|\leqslant\alpha\\ 0, & |x|>\alpha\end{cases}$$

with

$$c_\alpha^{-1}=\int_{-\infty}^{\infty}\exp\left(-\frac{\alpha^2}{\alpha^2-x^2}\right)\,\mathrm{d}x.$$

Definition 2. A sequence of integrable functions $\{f_k\}$ in a normed linear space $V(a,b)$ is said to converge to f relative to the class of test functions in $\mathscr{D}$ if $\{f_k\varphi\}$ and $\{f\varphi\}$ are in L_1 and

$$\lim\int_a^b f_k(x)\varphi(x)\,\mathrm{d}x=\int_a^b \lim f_k(x)\varphi(x)\,\mathrm{d}x$$
$$=\int_a^b f(x)\varphi(x)\,\mathrm{d}x,\qquad \forall\varphi\in\mathscr{D}.$$

Definition 3. A sequence $\{f_k\}\in V(a,b)$ is said to be Cauchy if for every $\varepsilon>0$ there exists integers J and K such that

$$\int_a^b (f_j-f_k)\varphi(x)\,\mathrm{d}x<\varepsilon,\qquad \text{whenever } j>J,\ k>K.$$

Since the set of real numbers $\int_a^b f_k(x)\varphi(x)\,\mathrm{d}x$ is complete, every Cauchy sequence $\{f_k\}\to f$ in the sense of definition 2.

Definition 4. A distribution F (or a generalised function) is an element of the completion of the linear metric space $V(a,b)$. That is, a distribution is an equivalence class of Cauchy sequences $\{f_k\}$, or the common limit of equivalent Cauchy sequences in $V(a,b)$.

An alternate definition of a distribution that follows from the above is that it is a continuous linear functional on $\mathscr{D}$ i.e. $F\in\mathscr{D}'$ such that

$$F_f(\varphi)=\lim\int_a^b f_k(x)\varphi(x)\,\mathrm{d}x,\qquad \forall\varphi\in\mathscr{D}.$$

The two most significant singular distributions in our work are the Dirac delta functional and the Principal Value distribution. Among the various equivalent Cauchy sequences in $\mathbb{R}$ of which these are the limits (Korevaar 1968), we choose

$$\delta_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}\tag{10a}$$

$$P_\varepsilon(x)\equiv\pi P_{0\varepsilon}(x)=\frac{x}{x^2+\varepsilon^2}\tag{10b}$$

to be the most relevant to our problem. Thus

$$\lim_{\varepsilon\to0}\int_{-\infty}^{\infty}\delta_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\int_{-\infty}^{\infty}\delta(x)\varphi(x)\,\mathrm{d}x,$$
$$\lim_{\varepsilon\to0}\int_{-\infty}^{\infty}P_\varepsilon(x)\varphi(x)\,\mathrm{d}x=⨍_{-\infty}^{\infty}\frac{\varphi(x)}{x}\,\mathrm{d}x$$
$$\equiv P\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}\,\mathrm{d}x,\qquad \forall\varphi\in\mathscr{D},$$

and from equation (8) it follows that

$$\frac{1}{x'-x} \Rightarrow P\frac{1}{x'-x} \pm \mathrm{i}\pi\delta(x'-x), \qquad z = x \pm \mathrm{i}\varepsilon.$$

We will examine the convergence properties of these distributions below.

Definition 5. For two Lebesgue integrable functions $f(x)$ and $g(x)$, $x \in \mathbb{R}$, their convolution $f(x) * g(x)$ is a well defined function and is given by

$$\begin{aligned} h(x) = f(x) * g(x) &= \int_{-\infty}^{\infty} f(x-x')g(x')\,\mathrm{d}x' \\ &= g(x) * f(x) = \int_{-\infty}^{\infty} g(x-x')f(x')\,\mathrm{d}x'. \end{aligned} \tag{11}$$

Now if $f_y \in L_p(\mathbb{R})$, it is easy to see that the following holds for the translated function $f_y(x) = f(x-y)$,

$$\|f_y\|_p = \|f\|_p.$$

A property of the convolution operator that follows from its definition and equation (11) is contained in the following theorem.

Theorem 1. If $f \in L_p$ and $g \in L_q$, $1 \leqslant p, q < \infty$, $p^{-1} + q^{-1} \geqslant 1$, then $f * g \in L_r$, $r^{-1} = p^{-1} + q^{-1} - 1$ and

$$\|f * g\|_r = \|f\|_p \|g\|_q.$$

In particular if $p^{-1} + q^{-1} = 1$, then $r = \infty$, and we have $\|f * g\|_\infty = \|f\|_p \|g\|_{p'}$, where p and p' are conjugate indices.

The equivalence class of equation (10*a*) is a particular case of the following characterisation of such classes.

Definition 6. If $\psi(x)$ is a non-negative integrable function over $\mathbb{R}$, if $\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 1$ and if $x\psi(x) \to 0$ as $|x| \to \infty$, then the sequence $\{\delta_\varepsilon(x)\} = \{\psi(x/\varepsilon)/\varepsilon\}$ is an equivalent Cauchy sequence for the delta functional $\delta(x)$, i.e., $\delta_\varepsilon \to \delta$ in $\mathscr{D}'$.

Equation (10*a*) is a specific case of definition 6 for $\psi(x) = (\pi(1+x^2))^{-1}$. More generally, we call a sequence of locally integrable functions $\{\psi_\varepsilon(x)\}$ on $\mathbb{R}$ a delta sequence of positive type if it satisfies the following conditions.

(i) $\int_{-A}^{A} \psi_\varepsilon(x)\,\mathrm{d}x \to 1$ as $\varepsilon \to 0$ for some finite constant A

(ii) For every constant $\alpha > 0$, $\psi_\varepsilon(x) \to 0$ uniformly for $\alpha \leqslant |x| < \infty$ as $\varepsilon \to 0$

(iii) $\psi_\varepsilon(x) \geqslant 0$ for all x and ε.

Thus while $\delta_\varepsilon(x)$ is a delta sequence of positive type, so is the sequence of equation (16) below, though this is not obtained from definition 6.

The convergence of distributions, being with respect to test functions, is not very useful in applications as the conditions imposed on the class of test functions are rather restrictive. Nevertheless, it is true that given an $\varepsilon > 0$, and a $f \in L_p$, there exist test functions arbitrarily close to f. To see this, consider a continuous function with compact support, $g \in C_0$, arbitrarily close to f, $|f(x) - g(x)| < \frac{1}{2}\varepsilon$. That such functions exist follows from the dominated convergence theorem. Now construct $\varphi_\alpha * g$, i.e., the function

$$\varphi_{1\alpha} = \int_{-\infty}^{\infty} \varphi_\alpha(x-x')g(x')\,\mathrm{d}x',$$

where φ_α is the test function of definition 1. Then $\varphi_{1\alpha} \in C_0^\infty$ and $|\varphi_{1\alpha} - g| < \frac{1}{2}\varepsilon$. Thus $|f - \varphi_{1\alpha}| < \varepsilon$, and the space of infinitely smooth functions with compact support is dense in L_p, $1 \leq p < \infty$. Definition 1 now completes the proof of the above assertion. From the definition of delta sequences, we may now state the following result for the convergence of the Poisson integral, $U(x, \varepsilon)$, relative to functions in L_p, and not necessarily in $\mathscr{D}$.

Theorem 2. If $\phi \in L_p$, $1 \leq p < \infty$, then $\phi_\varepsilon = \phi * \psi_\varepsilon$ tends to ϕ in L_p as $\varepsilon \to 0$. In particular, the Poisson integral of ϕ, $\phi_\varepsilon = \delta_\varepsilon * \phi = \int \delta_\varepsilon(x-x')\phi(x')\,\mathrm{d}x'$ tends to ϕ, i.e., $\|\phi - \phi_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$. The convergence is also both pointwise AE and uniform if ϕ is, in addition, continuous on every compact subinterval of $\mathbb{R}$.

Having considered the convergence properties of the Poisson integral, we now turn to the conjugate Poisson integral generated by the conjugate Poisson kernel, equation (10*b*). Define the Hilbert transform of $\phi \in L_p$, $1 \leq p < \infty$, by

$$H\phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(x')}{x'-x}\,\mathrm{d}x'$$

then the M Riesz theorem states the following.

Theorem 3. The Hilbert transform H is a bounded linear operator on $L_p(\mathbb{R})$, $1 < p < \infty$. In other words, given $\phi \in L_p(\mathbb{R})$, $1 < p < \infty$, $H\phi \in L_p(\mathbb{R})$, i.e., there exists a constant A_p independent of ϕ such that

$$\|H\phi\|_p^p = \int_{\mathbb{R}} |H\phi|^p\,\mathrm{d}x \leq A_p \int_{\mathbb{R}} |\phi|^p\,\mathrm{d}x. \qquad (12)$$

It must be noted however, that if $p = 1$, $H\phi$ does not necessarily belong to L_1. Nevertheless it is true that (Titchmarsh 1937)

$$\int_{\mathbb{R}} \frac{|H\phi|^p}{1+x^2}\,\mathrm{d}x < \infty, \qquad 0 < p < 1.$$

The following theorem states the nature of the convergence of $H_\varepsilon\phi = -P_{0\varepsilon} * \phi$ to $H\phi$.

Theorem 4. If $\phi \in L_p(\mathbb{R})$, $1 \leq p < \infty$, then $\{H_\varepsilon\phi\}$ tends to $H\phi$ in L_p as $\varepsilon \to 0$, i.e., $\|H\phi - H_\varepsilon\phi\|_p \to 0$ as $\varepsilon \to 0$. The convergence is also pointwise AE,

$$\lim_{\varepsilon \to 0} H_\varepsilon\phi(x) = H\phi(x), \qquad \text{AE}. \qquad (13)$$

Moreover, if $p > 1$, both $H_\varepsilon\phi$ and $H\phi$ satisfy the M Riesz theorem with the same constant A_p.

In passing we note that Plemelj formulae follow from theorem 2 and equation (13).

If instead of the whole real line, we restrict ourselves to a finite subinterval (a, b), then the basic property of the convolution integral, equation (11), no longer necessarily holds. For

$$\gamma(x) = \phi(x) * \psi(x) = \int_c^d \phi(x-x')\psi(x')\,dx'$$
$$= \int_{x-d}^{x-c} \phi(x')\psi(x-x')\,dx', \qquad a<x<b \tag{14}$$

and the last integral is not necessarily equal to

$$\int_c^d \phi(x')\psi(x-x')\,dx'.$$

If, however, we wish this to hold, that is if we wish that

$$\phi(x) * \psi(x) = \psi(x) * \phi(x), \qquad a<x<b; \qquad c<x'<d',$$

then one of the following conditions has to be satisfied.

(i) $\phi(x)$ and $\psi(x)$ are periodic functions with period $d-c$. (An application of this is to the formula for the partial sums of trigonometric Fourier series in $(-\pi, \pi)$ involving Dirichlet sequences.)

(ii) $x-c=d$ and $x-d=c$ that is $x=c+d$. Choose c and d to satisfy this equation for all $0 \leqslant x \leqslant A$, where A can be either finite or infinite. For example, we may take $c=0$ and $d=x$. (An example of this is the formula for the product of two Laplace transforms.)

(iii) If neither (i) nor (ii) holds, but $x-c=b$ and $x-d=a$, i.e., $b-a=d-c$ then with $a=c$, $b=d$ we have

$$\int_a^b \phi(x-x')\psi(x')\,dx' = \int_a^b \phi(x')\psi(x-x')\,dx' \tag{15}$$

and the convolution property is satisfied. In general, however, one can show that (Korevaar 1968) if the support of ψ is $[c, d]$, then a sufficient condition for the existence of the convolution on (a, b) is that $\phi(x)$ be integrable over $(a-d, b-c)$. Furthermore, if for $\psi(x)$ we consider a delta sequence of positive type, $\psi_\varepsilon(x)$, then

$$\phi_\varepsilon(x) = \int_a^b \phi(x-x')\psi_\varepsilon(x')\,dx', \qquad a<x<b$$
$$= \int_{-\infty}^{\infty} \phi(x-x')\psi_\varepsilon^*(x')\,dx'$$

where $\psi_\varepsilon^*(x) = \psi_\varepsilon(x)\chi[a, b]$, $\chi[a, b]$ being the characteristic function of (a, b), whenever the above conditions on ϕ and ψ are satisfied. On the finite interval, then, theorem 2 becomes the following.

Theorem 5. Let $\{\psi_\varepsilon\}$ be a delta sequence of positive type with supports in $[c, d]$ and let $[a, b]$ be any given interval. If $\phi \in L_p(a-d, b-c)$, $1 \leqslant p < \infty$, then the convolutions $\phi_\varepsilon = \psi_\varepsilon * \phi$ converge to ϕ in the L_p norm inside (a, b). This convergence is also uniform in every subinterval of (a, b) if ϕ is, in addition, continuous in (a, b), and pointwise in (a, b) if ϕ, beside being continuous on (a, b), belongs to $L_p(a, b)$.

In this work we will be concerned only with the case $d-c=b-a$, i.e., with condition (iii) above. Then (Korevaar 1968) theorem 5 implies the Weirstrass approximation theorem for the approximation of a continuous function in (a, b), as can be demonstrated by using the delta sequence

$$\psi_k^*(x)=\begin{cases} C_k^{-1}(A^2-x^2)^k, & |x|\geqslant A \\ 0, & |x|<A \end{cases} \tag{16}$$

$A\geqslant b-a$, with

$$C_k=\int_a^b (A^2-x^2)^k\,\mathrm{d}x,$$

and then $\psi_k\to\delta(x)$ as $k\to\infty$.

Because of equation (13), it follows that theorem 4 also has an equivalent version on a finite interval. This together with the Riesz Theorem, implies that the condition of Holder continuity for the density function $\phi(x)$ in SIE (7) can be replaced by L_p integrability, $1<p<\infty$.

In the definition of delta sequence, it is usual as in equation (16) above, to take

$$\int_a^b \psi_\varepsilon(x)\,\mathrm{d}x=1$$

where the integration is over the support of $\psi_\varepsilon(x)$. Thus

$$\delta_\varepsilon(x)=\frac{1}{\pi_\varepsilon}\frac{\varepsilon}{x^2+\varepsilon^2},\qquad a\leqslant x\leqslant b,$$

$$\pi_\varepsilon=\tan^{-1}(b/\varepsilon)-\tan^{-1}(a/\varepsilon).$$

i.e.,

$$\delta_\varepsilon(x)=\frac{1}{2\tan^{-1}(a/\varepsilon)}\frac{\varepsilon}{x^2+\varepsilon^2},\qquad -a\leqslant x\leqslant a \tag{17a}$$

and

$$\delta_\varepsilon(x)=\frac{1}{\tan^{-1}(a/\varepsilon)}\frac{\varepsilon}{x^2+\varepsilon^2},\qquad 0\leqslant x\leqslant a. \tag{17b}$$

We also use the notation

$$\delta_\varepsilon^*(x)=\begin{cases} \delta_\varepsilon(x), & x\in(a,b) \\ 0, & x\notin(a,b) \end{cases}$$

and

$$P_{0\varepsilon}(x)=\frac{1}{\pi}\frac{x}{x^2+\varepsilon^2},\qquad a\leqslant x\leqslant b.$$

A delta sequence not of the positive type but widely used in the theory trigonometric Fourier series in the Dirichlet sequence defined by

$$D_k^*(x)=\begin{cases} \dfrac{\sin(k+\frac{1}{2})x}{2\pi\sin\frac{1}{2}x}, & |x|\leqslant\pi \\ 0, & |x|>\pi \end{cases}$$

and $\lim_{k\to\infty} D_k^*(x)\to\delta(x)$ in the distributional sense of definition 2. The partial sums

$f_k(x)$ of the Fourier series for $f(x)$ can be shown to be given by

$$\begin{aligned} f_k(x) &= \int_{-\infty}^{\infty} f(x')D_k^*(x-x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} f(x-x')D_k^*(x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} f(x+x')D_k^*(x')\,\mathrm{d}x' \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} [f(x-x')+f(x+x')]D_k^*(x')\,\mathrm{d}x'. \end{aligned}$$

We do not need to go in the details of these sequences and the convergence characteristics of the $f_k(x)$ they generate but merely remark that in this case $f(x)$ must satisfy different, and more stringent, properties than those for the sequence of positive type mentioned above.

To end this section, we note, incidentally, that the difference between a finite and infinite region of integration in the solution of a singular integral equation is clearly demonstrated in the inverse Hilbert transform. Thus if

$$f(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{g(x')}{x'-x}\,\mathrm{d}x'$$

then the inverse transform is given by the symmetric formula

$$g(x) = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x'$$

but if

$$f(x) = \frac{1}{\pi} \fint_{-1}^{1} \frac{g(x')}{x'-x}\,\mathrm{d}x'$$

then for $g(z)$ to behave properly at ∓ 1 and vanish at infinity, one has the more complicated solution

$$g(x) = -\frac{1}{\pi} \fint_{-1}^{1} \left(\frac{1-x'^2}{1-x^2}\right)^{1/2} \frac{f(x')}{x'-x}\,\mathrm{d}x' + \frac{c}{\sqrt{1-x^2}}.$$

In the next section, we apply the present results to write down the rational function approximation of the singular eigenfunction, i.e., of equation (2*b*), as equation (19), and then apply the Case orthogonality integrals in the manner of Sengupta (1982) to obtain its coefficients.

3. The approximation problem

According to equation (3), any function Holder continuous in the extended sense can be expressed as

$$f(\mu) = a_{0+}\phi(\nu_0, \mu) + a_{0-}\phi(-\nu_0, \mu) + \int_{-1}^{1} A(\nu)\phi(\nu, \mu)\,\mathrm{d}\nu.$$

Inserting the expansion (2b) for $\phi(\nu, \mu)$ we get

$$g(\mu)=A(\mu)\lambda(\mu)+\frac{c}{2}\int_a^b \frac{\nu A(\nu)}{\nu-\mu}\,\mathrm{d}\nu \tag{18}$$

where $g(\mu)$ on the LHS is the function

$$f(\mu)-a_{0+}\phi(\nu_0,\mu)-a_{0-}\phi(-\nu_0,\mu)$$

and is also Holder continuous, either on $(a, b)=(-1, 1)$ or on $(0, 1)$ when $a_{0-}=0$. If $A(\nu)\in L_p(a, b)$ also, then from theorems 3 and 4 applied to the finite interval (a, b), $H(\nu A)\in L_p(a, b)$, $1<p<\infty$, and

$$H(\nu A)=\lim_{\varepsilon\to 0} H_\varepsilon(\nu A)$$
$$=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_a^b \frac{\nu-\mu}{(\nu-\mu)^2+\varepsilon^2}\,\nu A(\nu)\,\mathrm{d}\nu.$$

Hence the SIE has a solution for $g(\mu)\in L_p$, $1<p<\infty$, and therefore equation (3) is valid when $f(\mu)\in L_p$, $1<p<\infty$. In fact we have the general result (Titchmarsh 1937) that when $\phi(x)\in L_p(R)$, $1<p<\infty$, and also satisfies a Holder condition on $\mathbb{R}$ with index $\alpha<1$, then $H\phi$ (which is in $L_p(\mathbb{R})$ by theorem 3) is also Holder continuous with the same index α. It is to be observed that the result fails to hold when $\alpha=1$.

In view of the remarks of § 1, we now introduce a rational fraction approximation for $\phi(\nu, \mu)$, $\nu\in(-1, 1)$, that tends to the RHS of equation (2b) as $\varepsilon\to 0$. Following the basic ideas introduced in Sengupta (1982) (we shall henceforth refer to this work as I), we use the orthogonality expressions satisfied by $\{\phi(\pm\nu_0, \mu), \phi(\nu, \mu)\}$ to determine the coefficients of the rational function approximant of $\phi(\nu, \mu)$. There are three such independent integrals, (I.15), (I.16) and (I.17) for the full range case, and two for half range, equations (I.7a,d) besides equations (I.6a) and (I.7e) that completely specify the problems. Based on the discussions of § 2, we can conclude that the proper [1/2] rational fraction approximation $\phi_\varepsilon(\nu, \mu)$ for $\phi(\nu, \mu)$, $\nu\in(-1, 1)$, is

$$\phi_\varepsilon(\nu,\mu)=\frac{c\nu}{2}\frac{\nu-\mu}{(\nu-\mu)^2+\varepsilon^2}+\frac{\lambda_\varepsilon(\nu)}{\pi_\varepsilon(\nu)}\frac{\varepsilon}{(\nu-\mu)^2+\varepsilon^2},\qquad \nu\in(-1,1) \tag{19}$$

which tends, as $\varepsilon\to 0$, to the sum of the two distributions that comprise $\phi(\nu, \mu)$. (This statement on the convergence of ϕ_ε is made more precise below.) Approximation (19) can be used, because of theorems 2 and 4 restricted to the finite interval, in equations (I.15), (I.16) or (I.7a). However equations (I.17) or (I.7d) are not covered by these theorems. To see how these equations can be used, we note that they can be written as the convolutions $\omega(\mu)\phi_\varepsilon(\nu, \nu-\mu) * \phi_\varepsilon(\nu', \mu)$ where (Case and Zweifel 1967, I)

$$\omega(\mu)=\begin{cases}\mu, & -1\leqslant\mu\leqslant 1\\ W(\mu), & 0\leqslant\mu\leqslant 1\end{cases}$$

$f(t-a) * g(t)$ being defined as $\int f(\tau-a)g(t-\tau)\,\mathrm{d}\tau$. This convolution converges, as $\varepsilon\to 0$, to $\int\omega(\mu)\phi(\nu, \mu)\phi(\nu', \mu)\,\mathrm{d}\mu$ distributionally. This is a consequence of the result (Korevaar 1968) that if f_ε and g_ε are two distributions such that $f_\varepsilon\to f$ and $g_\varepsilon\to g$, then $f_\varepsilon * g_\varepsilon$ exists as a distribution if either f_ε or g_ε has bounded support and then $f_\varepsilon * g_\varepsilon\to f * g$. (This, for example, allows us to write $\delta(\nu-\mu) * \delta(\mu)=\int\delta(\nu-\mu)\delta(\mu-\nu')\,\mathrm{d}\mu=\delta(\nu-\nu')$). Therefore the above, together with the results of § 2 ensures that if $\phi_\varepsilon(\nu, \mu)$ is given by equation (19), then equations (I.15), (I.16) and (I.17) or equations (I.7a) and

(I.7d) can be used to obtain the unknown coefficients, and in this case we also have

$$\|\phi(\cdot,\mu)-\phi_\varepsilon(\cdot,\mu)\|_p \to 0, \tag{20a}$$

and

$$\phi_\varepsilon(\cdot,\mu)\to\phi(\cdot,\mu), \qquad \text{pointwise AE} \tag{20b}$$

as $\varepsilon \to 0$. Of course, it is to be understood that the convergence implied by equations (20a) and (20b) is in the sense of the theorems of § 2.

Using the normalisation of ϕ_ε, equation (2d), we get from equation (19),

$$\lambda_\varepsilon(\nu)=\frac{\pi_\varepsilon}{\tan^{-1}(1+\nu)/\varepsilon+\tan^{-1}(1-\nu)/\varepsilon}\left(1-\frac{c\nu}{4}\ln\frac{(1+\nu)^2+\varepsilon^2}{(1-\nu)^2+\varepsilon^2}\right) \tag{21}$$

where $\pi_\varepsilon = 2\tan^{-1} 1/\varepsilon$ and we have $\lambda_\varepsilon \to \lambda$. For a given non-zero choice of ε, there are two parameters in equation (19) that can be used to fit $\phi_\varepsilon(\nu,\mu)$ to $\phi(\nu,\mu)$. One of these is obviously the desired discrete $\{\nu_j\}$—as $\varepsilon\to 0$, $\{\nu_j\}\to(-1,1)$ or $(0,1)$ as the case may be—while the other is $c_\varepsilon \to c$. We can use any two of the three orthogonality integrals for the full range problem, the equation omitted ensuring that there exists a solution for the c_ε and $\{\nu_j\}$, $\varepsilon > 0$†, while in the half range case any one of the two systems of two equations each, namely either (I.7a) and (I.7d) with $N(\nu)$ on the RHS replaced by the obvious approximation

$$N_\varepsilon(\nu)=\nu[\lambda_\varepsilon^2(\nu)+\tfrac{1}{4}\pi^2c_\varepsilon^2\nu^2],$$

or equation (I.7a) and one of the various integrals that may be derived—e.g., equation (I.7b) with the actual c in its RHS—can be used. Any of these choices will give a valid approximate solution. Obviously, in equations (I.17) or (I.7d) we use $\delta_\varepsilon(\nu-\nu')$ on their RHS. Once the c_ε and $\{\nu_j\}$ have been determined the $f(\mu)\in L_p$, $1<p<\infty$, and the solution of the neutron transport problem, can be written as the desired finite sum (I).

4. Discussions

In this section, we give an account of some methods for obtaining the coefficients of a rational function approximation so as to be able to gain a perspective of our method of § 3.

Let $f(x)$ be a continuous function in (a, b) and let $P_M(x)/Q_N(x)$ be a rational fraction approximation to f in $a\leq x\leq b$ such that

$$\frac{P_M(x)}{Q_N(x)}=\frac{a_0+a_1x+\cdots+a_Mx^M}{1+b_1x+\cdots+b_Nx^N}$$

Then the coefficients $\{a_i\}_0^M$, $\{b_i\}_1^N$ may be obtained by any one of the following methods.

(i) Consider $f(x)Q_N(x)-P_M(x)$. Expand $f(x)$ in a Taylor series in x (or, in a series of orthogonal polynomials $\pi_i(x)$, in which case both P_M and Q_N are to be expressed in terms of the same set of polynomials), multiply the series for f and Q_N, and equate the first $M+N+1$ coefficients of $fQ-P$ to zero. This means that the difference $R_{M,N}(x)=fQ_N-P_M=O(x^{M+N+1})$. If orthogonal polynomials are used, then

† If all the three equations are used in the determination of the constants, then the only possible solution is $c_\varepsilon = c$, $\{\nu_j\}=(-1,1)$ and $\varepsilon=0$.

it must be possible to write $\pi_i \pi_j = \Sigma_k A_{ijk} \pi_k$ and have $R_{M,N}(x) = O(\pi_{M+N+1})$. This procedure gives the standard $[M/N]$ Padé approximation to $f(x)$.

(ii) Choose the coefficients such that the least square error of $fQ_N - P_M$ is minimised. Then the coefficients are the solution of the set of equations

$$\int_a^b \omega(x)(fQ_N - P_M)\pi_i(x)\,dx = 0, \qquad i = 0, 1, \ldots, M+N,$$

where the $\pi_i(x)$ are either orthogonal polynomials or x^i, and $\omega(x)$ is a suitable weight function. If orthogonal polynomials are used, then (i) and (ii) give the same solution i.e., $\{a_i\}_0^M$, $\{b_i\}_1^N$, are the solutions of the $M+N+1$ equations

$$(fQ_N - P_M, \pi_i) = 0, \qquad i = 0, 1, \ldots, M$$

$$(fQ_N, \pi_i) = 0, \qquad i = M+1, \ldots, M+N$$

while the coefficients of $R_{M,N}(x)$ are given by $(fQ_N, \pi_i) = 0$, $i = M+N+1, \ldots$. On the other hand, if $\pi_i(x) = x^i$, then (i) leads to the set of $M+N+1$ equations

$$a_j = \sum_{i=0}^{j} c_i b_{j-i}, \qquad j = 0, 1, \ldots, M$$

$$0 = \sum_{i=j-n}^{j} c_i b_{j-1}, \qquad j = M+1, \ldots, M+N$$

$\{c_i\}$ being the expansion coefficients of $f(x)$, and (ii) to the equations

$$(fQ_N - P_M, x^i) = 0, \qquad i = 0, 1, \ldots, M+N$$

for the $\{a_i\}$ and $\{b_i\}$. In both (i) and (ii) the error is $f - P_M/Q_N = R_{M,N}/Q_N$ where $fQ_N - P_M = R_{M,N}$.

(iii) As a variant of (ii) we may determine the coefficients from the nonlinear equations

$$\int_a^b \omega(x)\left(f(x) - \frac{P_M(x)}{Q_N(x)}\right)\pi_i(x)\,dx = 0, \qquad i = 0, 1, \ldots, M+N$$

so that $R_{M,N} = f - P_M/Q_N$.

The formulation of § 3 is an adaption of (iii) while that suggested in (I), i.e., a constrained Padé approximation, is a combination of (i) and (iii). The present method, as compared with that of (I) is simpler and more pertinent to our problem.

As a comparison of the different approaches above, consider the rational approximation of exp x in terms of x. Then we have for the [1/1] approximant,

$$[1/1] = \frac{a_0 + a_1 x}{1 + b_1 x},$$

the following sets of equations for a_0, a_1, b_1 in the interval (0, 2).

(i)(a) Padé approximation

$$a_0 = 1$$

$$a_1 = 1 + b_1$$

$$0 = \tfrac{1}{2} + b_1.$$

(b) Constrained Padé approximation with constraint $m_0 = \int_0^2 (a_0 + a_1 x)/(1 + b_1 x)\, dx$,

$$a_0 = 1$$

$$a_1 = 1 + b_1$$

$$b_1 = \tfrac{1}{2}\{\exp[b_1^2(2/b_1 - m_0 + 2)] - 1\}.$$

(ii) Minimising error of $fQ_1 - P_1$ in $L_2(0, 2)$,

$$2a_0 + 2a_1 - m_1 b_1 = m_0$$

$$2a_0 + \tfrac{8}{3}a_1 - m_2 b_1 = m_1$$

$$\tfrac{8}{3}a_0 + 4a_1 - m_3 b_1 = m_2.$$

(iii) Moments of f and P_1/Q_1 equal

$$2a_0 + 2a_1 - m_1 b_1 = m_0$$

$$\tfrac{8}{3}a_1 - m_2 b_1 = m_1 - 2a_0$$

$$a_0 = \frac{1}{\ln(1 + 2b_1)}(m_0 b_1 - 2a_1) + \frac{a_1}{b_1}$$

where

$$m_i = \int_0^2 e^x x^i \, dx.$$

To solve the above sets of equations, one can proceed as follows. Iterate for b_1 in (i) (b) and then solve for a_0 and a_1. In (iii), the three nonlinear equations can, by proper elimination, be reduced to the two linear and one nonlinear equations shown. The two linear equations are the same as the corresponding equations of (ii). To solve for the set (iii) therefore, use a_1, b_1 as obtained in (ii) as the first iterate in the third equation of (iii) and use this a_0 for obtaining revised values of a_1 and b_1 from the first two equations. These are then used in the third equation to get an improved value of a_0 and the process continued till proper convergence is obtained.

The respective value of the coefficients for $\exp(x)$ are

(i) (a)	$a_0 = 1$,	$a_1 = 0.5$,	$b_1 = -0.5$
(b)	$a_0 = 1$,	$a_1 = 0.632\,610$,	$b_1 = -0.367\,390$
(ii)	$a_0 = 0.955\,215$,	$a_1 = 0.903\,751$,	$b_1 = -0.318\,406$
(iii)	$a_0 = 0.945\,516$,	$a_1 = 0.930\,482$,	$b_1 = -0.314\,345$.

This sample calculation shows immediately the advantage of using integral constraints as compared to standard Padé approximation: no unusual behaviour of the approximant, like the pole at $x = 2$ occurs. This is true for all orders of approximation and all finite intervals, and provides the assurance of our method.

We conclude this paper with the following comments. The present method shows that any function in L_p, $p > 1$, can be expressed as equation (3), and hence has an approximation with respect to the basis set constructed from the discretisation of ν.

Expansions of generalised functions, like the Dirac delta, in terms of the Case eigenfunctions can be understood from the discussions in §§ 2 and 3. Though equation (19) is a [1/2] approximant, there are only two parameters to be determined, as compared to the usual four, because of their combination in equation (19). The fundamental result on which this paper is based is the convergence of the Poisson and conjugate Poisson integrals $U(x, \varepsilon)$ and $V(x, \varepsilon)$ as $\varepsilon \to 0$, which suggests a new approximate method of solving singular integral equations, for some non-zero ε, by converting it to a Fredholm integral equation. Thus, the SIE (7),

$$\alpha(x)\phi(x)+\beta(x)\frac{1}{2\pi i}\int_{-1}^{1}\frac{\phi(x')}{x'-x}\,dx'=B(x)$$

can be replaced by the approximately equivalent Fredholm integral equation

$$\alpha(x)\phi_\varepsilon(x)+\beta(x)\frac{1}{2\pi i}\int_{-1}^{1}\frac{(x'-x)\phi_\varepsilon(x')}{(x'-x)^2+\varepsilon^2}\,dx'=B(x), \qquad \varepsilon>0,$$

and because of § 2, we must have

$$\phi_\varepsilon(x)\to\phi(x), \qquad \varepsilon\to 0$$

the convergence being in accordance with the theorems stated there. The results of our investigation into this approach for the approximate solution of SIE will be presented subsequently. And finally, on a point of emphasis, equation (19) is our rational function approximation to Case's singular eigenfunction, equation (2*b*).

References

Case K M and Zweifel P F 1967 *Linear Transport Theory* (Reading, Mass: Addison Wesley)
Dow M L and Elliot D 1979 *SIAM J. Numer. Anal.* **16** 115
Elliot D 1982 *SIAM J. Numer. Anal.* **19** 816
Korevaar J 1968 *Mathematical Methods* Vol I (New York: Academic)
Larsen E W and Habetler G J 1973 *Commun. Pure Appl. Math.* **26** 525
Roos B W 1969 *Analytic Functions and Distributions in Physics and Engineering* (New York: Wiley)
Sadosky C 1979 *Interpolation of Operators and Singular Integrals* (New York: Marcel Dekker)
Sengupta A 1982 *J. Phys. A: Math. Gen.* **15** L487
Titchmarsh E C 1937 *Introduction to the theory of Fourier Integrals* (Oxford: OUP)